

Loop Algebra Moment Maps and Hamiltonian Models for the Painlevé Transcendents

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Abstract

The isomonodromic deformations underlying the Painlevé transcendents are interpreted as nonautonomous Hamiltonian systems in the dual $\tilde{\mathfrak{g}}_R^*$ of a loop algebra $\tilde{\mathfrak{g}}$ in the classical R -matrix framework. It is shown how canonical coordinates on symplectic vector spaces of dimensions four or six parametrize certain rational coadjoint orbits in $\tilde{\mathfrak{g}}_R^*$ via a moment map embedding. The Hamiltonians underlying the Painlevé transcendents are obtained by pulling back elements of the ring of spectral invariants. These are shown to determine simple Hamiltonian systems within the underlying symplectic vector space.

Dedicated to Jerrold Marsden on the occasion of his 50th birthday.

Introduction

The Painlevé transcendents $P_I - P_{VI}$ have been interpreted as isomonodromic deformation equations for first order matrix systems $\partial\psi/\partial\lambda = \mathcal{N}(\lambda)\psi$, where $\mathcal{N}(\lambda)$ is rational in λ , in [FN, JM]. The Hamiltonian structure of these equations has been studied by many authors (see e.g. [Ok] and references therein). In this paper, the Hamiltonians for $P_I - P_{VI}$ are derived within the framework of the Adler-Kostant-Symes (AKS) theorem and the classical R -matrix approach [S] on a rational coadjoint orbit in the dual $\tilde{\mathfrak{g}}^*$ of a loop algebra $\tilde{\mathfrak{g}}$. This construction allows for the parametrization of the underlying phase space by Darboux coordinates on a finite-dimensional symplectic vector space M via a moment map embedding, according to a general scheme developed in [AHP, AHH1, AHH2]. The AKS Hamiltonians for $P_I - P_{VI}$ on $\tilde{\mathfrak{g}}^*$ pull back to M where they may be interpreted as simple mechanical systems. In each case the Painlevé transcendent is obtained by Hamiltonian reduction. This scheme is used in [H] to parametrize the phase spaces for P_V and P_{VI} . In [HTW] a special case of P_{II} (and P_{IV}) arising in the spectral theory of random matrices is given a similar loop algebra formulation. In the present work we give a complete list of Hamiltonian systems related to the Painlevé transcendents obtained within this framework.

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1. Darboux Coordinates on Rational Coadjoint Orbits in Loop Algebras.

1a. Loop algebras. Let \mathfrak{g} be a Lie algebra. For the purpose of this paper it is sufficient to consider $\mathfrak{g} = \mathfrak{gl}(2)$. Let \mathcal{C} be a closed, simple curve in $\mathbf{P}^1(\mathbf{C})$ (typically, a circle centered at the origin), dividing it into an interior region U^+ , containing 0 and an exterior region U^- , containing ∞ . The space $\tilde{\mathfrak{g}}$ of smooth maps $X : \mathcal{C} \rightarrow \mathfrak{g}$ with the natural Lie algebra structure induced by that on \mathfrak{g} defines our loop algebra. We consider the splitting of $\tilde{\mathfrak{g}}$ as a vector space direct sum

$$\tilde{\mathfrak{g}} = \mathfrak{g}^+ \oplus \mathfrak{g}^- \quad (1.1)$$

of the subalgebra \mathfrak{g}^+ , consisting of maps admitting a holomorphic extension to U^+ and \mathfrak{g}^- , consisting of maps X admitting a holomorphic extension to U^- , the latter normalized by the condition $X(\infty) = 0$. On $\tilde{\mathfrak{g}}$ we define the Ad-invariant inner product

$$\langle X, Y \rangle = \frac{1}{2\pi i} \oint_{\mathcal{C}} \text{tr}(X(\lambda)Y(\lambda))d\lambda, \quad (1.2)$$

which provides an identification of $\tilde{\mathfrak{g}}$ with a dense subspace of $\tilde{\mathfrak{g}}^*$. In the dual decomposition

$$\tilde{\mathfrak{g}}^* = \mathfrak{g}^{+*} \oplus \mathfrak{g}^{-*}, \quad (1.3)$$

we have the corresponding identifications of \mathfrak{g}^{\pm} with dense subspaces of $(\mathfrak{g}^{\mp})^*$.

Let $P_{\pm} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^{\pm}$ be the projections to these subalgebras, and define

$$R := \frac{1}{2}(P_+ - P_-). \quad (1.4)$$

Then the bracket defined by

$$[X, Y]_R = [R(X), Y] + [X, R(Y)] \quad (1.5)$$

determines a second Lie algebra structure on $\tilde{\mathfrak{g}}$, and R is referred to as a “classical R -matrix”. We shall write $\tilde{\mathfrak{g}}_R$ when referring to $\tilde{\mathfrak{g}}$ endowed with the Lie bracket (1.5) and denote by Ad_R^* the coadjoint representation with respect to this modified Lie bracket, while Ad^* continues to denote the ordinary coadjoint representation.

On $\tilde{\mathfrak{g}}_R^*$ we have the Lie-Poisson structure given by

$$\{f, g\}_R(\mu) = \mu([df(\mu), dg(\mu)]_R), \quad f, g \in C^\infty(\tilde{\mathfrak{g}}_R^*), \mu \in \tilde{\mathfrak{g}}_R^*. \quad (1.6)$$

(The Lie Poisson structure with respect to the original Lie bracket on $\tilde{\mathfrak{g}}$ plays no direct role in what follows.)

1b. Moment maps into loop algebras. Let $M = \mathbf{R}^{2N}$ be a finite-dimensional symplectic vector space with symplectic form Ω given by

$$\Omega = \sum_{i=1}^n dx_i \wedge dy_i. \quad (1.7)$$

The Poisson space $\tilde{\mathfrak{g}}_R^*$ contains finite dimensional Poisson subspaces with symplectic leaves consisting of orbits whose elements are rational in λ . Let $\mathfrak{g}_A \subset \tilde{\mathfrak{g}}_R^*$ be such a finite-dimensional Poisson-subspace. A Poisson map

$$J : M \rightarrow \mathfrak{g}_A \subset \tilde{\mathfrak{g}}_R^* \quad (1.8)$$

whose coefficients generate complete flows may be interpreted as an equivariant moment map for a suitable group action. A general scheme for the construction of such moment maps into loop algebras with applications to ODE's and PDE's can be found in [AHP, AHH1, AHH2, HW].

The image of a moment map of type (1.8) consists of elements of the form

$$\mathcal{N}(\lambda) = \lambda^{n_0} Y + \sum_{l_0=0}^{n_0-1} N_{0,l_0} \lambda^{l_0} + \sum_{i=1}^n \sum_{l_i=1}^{n_i} \frac{N_{i,l_i}}{(\lambda - \alpha_i)^{l_i}}, \quad (1.9)$$

where the poles α_i are viewed as parameters characterizing the Ad_R^* -orbits, $Y \in \mathfrak{g}$ is a constant element whose components are Casimir invariants on \mathfrak{g}_A and the entries of $N_{k,l} \in \mathfrak{g}$ are typically rational expressions in (x_i, y_i) .

In the examples below it will be possible to choose J such that it is locally a symplectic diffeomorphism onto a coadjoint orbit \mathcal{O} in \mathfrak{g}_A , thereby parametrizing \mathcal{O} in terms of the Darboux coordinates on M .

1c. Autonomous and nonautonomous Hamiltonian systems in loop algebras and isomonodromic deformation equations. Let \mathcal{I} be the ring of Ad^* -invariant polynomial functions on $\tilde{\mathfrak{g}}_R^*$, restricted to \mathfrak{g}_A . These functions will also be called *spectral invariants*. The Adler-Kostant-Symes (AKS) theorem, in its generalized R -matrix form, [S] tells us that

- (1) elements of \mathcal{I} Poisson-commute (and hence so do their pullbacks under the Poisson map J).
- (2) Hamilton's equations for $H \in \mathcal{I}$ are given by

$$\frac{d\mathcal{N}}{dt} = [P_\sigma \circ dH(\mathcal{N}), \mathcal{N}], \quad \mathcal{N} \in \mathfrak{g}_A, \quad (1.10)$$

where

$$P_\sigma = (1 + \sigma)P_+ + \sigma P_-, \quad \sigma \in \mathbf{R}. \quad (1.11)$$

(The equivalence of this equation for all values of $\sigma \in \mathbf{R}$ follows from the Ad^* -invariance of H .)

These autonomous systems therefore preserve the spectrum of $\mathcal{N}(\lambda)$, and the ring of invariants is identified with the coefficients of the characteristic equation

$$\det(\mathcal{N}(\lambda) - \zeta \mathbf{I}) = 0 \quad (1.12)$$

defining the “spectral curve” X . Identifying one or more of the α_i 's or the entries of Y with the flow-parameter t , the resulting nonautonomous system is given by

$$\frac{\partial \mathcal{N}}{\partial t} = [P_\sigma \circ dH(\mathcal{N}), \mathcal{N}] + \mathcal{N}_t, \quad \mathcal{N} \in \mathfrak{g}_A, \quad (1.13)$$

where \mathcal{N}_t denotes the partial derivative with respect to the explicit dependance on t . This may still be viewed as a Hamiltonian system on M , induced by the nonautonomous Hamiltonian $J^*(H)$. The system (1.13) is no longer isospectral, but defines an isomonodromic deformation equation if the condition

$$\frac{\partial \mathcal{N}}{\partial t} = \frac{\partial}{\partial \lambda}(P_\sigma \circ dH(\mathcal{N})) \quad (1.14)$$

is satisfied. The nonautonomous system (1.13) is then equivalent to the commutativity

$$[\mathcal{D}_\lambda, \mathcal{D}_t] = 0 \quad (1.15)$$

of the differential operators

$$\mathcal{D}_\lambda = \frac{\partial}{\partial \lambda} - \mathcal{N}(\lambda) \quad (1.16a)$$

$$\mathcal{D}_t = \frac{\partial}{\partial t} - P_\sigma \circ dH(\mathcal{N})(\lambda). \quad (1.16b)$$

An important class of Hamiltonian systems on \mathbf{g}_A leading to systems of differential operators defining isomonodromic deformation equations is obtained if Y is diagonalizable, $n_0 = 0$ and the poles in (1.9) are simple. (See [H] for details).

1d. Reductions. From now on we specialize to the case $r = 2$ in order to simplify the exposition. This is the case relevant for the Painlevé transcendents. The elements of \mathcal{I} are invariant under the action of the (generically 1-dimensional) isotropy group $H_Y \subset Gl(2)$ of Y (quotienting by the centre to obtain an effective action). It follows that the spectral invariants project to the Marsden-Weinstein reduction of \mathcal{O} by H_Y . These reductions will be shown to give the phase spaces underlying the Painlevé transcendents $P_I - P_V$. For P_{VI} the image of J is entirely included in $\tilde{\mathfrak{g}}^{+*}$, so that $Y = 0$, and therefore $H_Y = Sl(2)$. In this case the “unreduced” phase space is obtained by partially fixing the moment map J_{H_Y} generating the $Sl(2)$ -action. The corresponding level sets will be chosen so as to define a symplectic submanifold of \mathcal{O} .

1e. Canonical spectral coordinates. A coadjoint orbit consisting of elements of the form (1.9) may alternatively be parametrized by “spectral Darboux coordinates” $(\ln(w), a, u_i, v_i)$, $i = 1, \dots, g$, where g is the genus of the spectral curve X . In the autonomous, isospectral case, these lead to a complete separation of variables on generic coadjoint orbits, and thereby a linearization of the flows of the Jacobi variety of X (cf. [AHH3]). Taking $Y_{21} = 0$ they are defined by the equations

$$\mathcal{N}(\lambda)_{21} = \frac{w \prod_{i=1}^g (\lambda - u_i)}{\tilde{a}(\lambda)}, \quad (1.17)$$

$$\mathcal{N}(u_i)_{22} = v_i,$$

and $\tilde{a}(\lambda) = \prod_{i=1}^n (\lambda - \alpha_i)^{n_i}$. To complete the system we must add the generator a of the 1-dimensional group H_Y , which is canonically conjugate to the “ignorable” coordinate w . The resulting symplectic form on \mathcal{O} , before the reduction by H_Y , is given by

$$\omega = \sum_{i=1}^g du_i \wedge dv_i + d \ln(w) \wedge da. \quad (1.18)$$

(In certain cases, it is convenient to make a further canonical coordinate transformation $v_i \rightarrow v_i + f_i(u_i)$ to simplify the resulting expressions.) Choosing a level set for a ; that is, performing the Marsden-Weinstein reduction by H_Y , we obtain a symplectic manifold with reduced symplectic form

$$\omega_{\text{red}} = \sum_{i=1}^g du_i \wedge dv_i. \quad (1.19)$$

For the examples given below, we always have $g = 1$, and $u_1 =: u$ is, essentially, the variable satisfying the Painlevé equation.

2. Hamiltonian Models for the Painlevé Transcendents.

Painlevé I. Let $M = \mathbf{R}^4$, with symplectic form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \quad (2.1)$$

The moment map $J_I : \mathbf{R}^4 \rightarrow \tilde{\mathfrak{g}}_R^*$ is defined by

$$J_I(x_1, x_2, y_1, y_2) := \mathcal{N}(\lambda) = \lambda^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 & x_2 \\ \kappa & -x_1 \end{pmatrix} + \begin{pmatrix} -y_2 + x_1 x_2 & y_1 + t/2 \\ -x_1^2 - \kappa x_2 & y_2 - x_1 x_2 \end{pmatrix}, \quad (2.2)$$

where κ is a nonzero constant and t is the deformation parameter (time). The isotropy group H_Y is

$$H_Y = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbf{R} \right\}. \quad (2.3)$$

Its action on M is generated by

$$a := \kappa(y_1 - x_2^2) - 2x_1 y_2 + x_1^2 x_2. \quad (2.4)$$

This case is exceptional, in that the equations (1.17) do not provide the coordinate $\ln(w)$ conjugate to a , but rather, a constant $w = \kappa$. Nevertheless, they yield the coordinates (u, v) on the reduced phase space,

$$u = \kappa^{-1} x_1^2 + x_2, \quad v = y_2 - x_1 x_2 - \kappa^{-1} x_1^3 \quad (2.5)$$

and we may choose, e.g.

$$z := \frac{x_1}{\kappa} \quad (2.6)$$

as the remaining canonical coordinate. The symplectic form is then given by

$$\omega = du \wedge dv + dz \wedge da \quad (2.7)$$

and the reduced form is

$$\omega_{\text{red}} = du \wedge dv. \quad (2.8)$$

The Hamiltonian \mathcal{H}_I on the unreduced manifold is given in terms of (x_i, y_i) by

$$\mathcal{H}_I(x_1, y_1, x_2, y_2, t) = \frac{1}{4} \text{res}_{\lambda=0} \text{tr}(\lambda^{-1} \mathcal{N}^2(\lambda)) = \frac{1}{2} [(y_2 - x_1 x_2)^2 - (x_1^2 + \kappa x_2)(y_1 + t/2)]. \quad (2.9)$$

The differential operator \mathcal{D}_t in (1.16b) defining the isomonodromic deformation of \mathcal{D}_λ corresponding to \mathcal{H}_I is obtained by choosing $\sigma = 0$ in (1.16b), giving

$$\mathcal{D}_t = \frac{\partial}{\partial t} - \lambda \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} x_1 & x_2 \\ \kappa & -x_1 \end{pmatrix}. \quad (2.10)$$

On the Marsden-Weinstein reduced space corresponding to the invariant level set $a = a_0$ the reduced Hamiltonian $\tilde{\mathcal{H}}_I$ is given in terms of (u, v) by

$$\tilde{\mathcal{H}}_I(u, v, t) = \frac{v^2}{2} - \frac{\kappa u^3}{2} - \frac{\kappa u t}{4} - \frac{a_0 u}{2}. \quad (2.11)$$

Hamilton's equations for $\tilde{\mathcal{H}}_I$ imply

$$\ddot{u} = \frac{3\kappa}{2} u^2 + \frac{\kappa}{4} t + \frac{a_0}{2}. \quad (2.12)$$

This gives the standard form of P_I (cf. [I]) for $\kappa = 4$ and $a_0 = 0$.

Painlevé II. Let $M = \mathbf{R}^4$ again with standard symplectic form (2.1). The moment map J_{II} into $\tilde{\mathfrak{g}}_R^*$ is defined by

$$J_{II}(x_1, x_2, y_1, y_2) := \mathcal{N}(\lambda) = \lambda^2 \begin{pmatrix} \frac{\kappa}{2} & 0 \\ 0 & -\frac{\kappa}{2} \end{pmatrix} + \lambda \begin{pmatrix} 0 & -\kappa y_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} x_2 y_1 + \frac{t}{2} & -\kappa y_2 \\ x_1 & -x_2 y_1 - \frac{t}{2} \end{pmatrix}, \quad (2.13)$$

where κ is a nonzero constant and t is the deformation parameter. The isotropy group H_Y is

$$H_Y = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \mid b \in \mathbf{R}^* \right\}. \quad (2.14)$$

Its action on M is generated by

$$a := x_1 y_1 + x_2 y_2. \quad (2.15)$$

The spectral Darboux coordinates on the unreduced phase space are given by

$$u = -x_1/x_2, \quad v = -x_2 y_1, \quad w = -x_2, \quad \text{and} \quad a = x_1 y_1 + x_2 y_2. \quad (2.16)$$

In terms of these, the symplectic form is

$$\omega = du \wedge dv + d \ln(w) \wedge a. \quad (2.17)$$

The Hamiltonian \mathcal{H}_{II} on M is given by

$$\mathcal{H}_{II}(x_1, y_1, x_2, y_2, t) = \frac{1}{2\kappa} \text{res}_{\lambda=0} \text{tr}(\lambda^{-1} \mathcal{N}^2(\lambda)) = \frac{1}{\kappa} ((x_1 y_2)^2 + t x_2 y_1 + t^2/4 - \kappa x_1 y_2). \quad (2.18)$$

The differential operator \mathcal{D}_t in (1.16b) defining the isomonodromic deformation of \mathcal{D}_λ corresponding to \mathcal{H}_{II} is again obtained by setting $\sigma = 0$ in (1.16b), giving

$$D_t = \frac{\partial}{\partial t} - \lambda \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 & -y_1 \\ \kappa^{-1} x_2 & 0 \end{pmatrix}. \quad (2.19)$$

On the level set $a = a_0$, the reduced Hamiltonian $\tilde{\mathcal{H}}_{II}$ is expressed in terms of (u, v) by

$$\tilde{\mathcal{H}}_{II}(u, v) = \frac{1}{\kappa} (v^2 - vt + t^2/4 - \kappa u(uv - a_0)), \quad (2.20)$$

and the reduced symplectic form ω_{red} is again

$$\omega_{\text{red}} = du \wedge dv. \quad (2.21)$$

Hamilton's equations for $\tilde{\mathcal{H}}_{II}$ are equivalent to

$$\ddot{u} = 2u^3 + 2\kappa^{-1}t u + \alpha, \quad (2.22)$$

where $\alpha = -\kappa^{-1}(2a_0 + 1)$. Setting $\kappa = 2$, (2.22) gives the standard form of P_{II} (cf. [I]).

Painlevé III. Again, $M = \mathbf{R}^4$ with symplectic form (2.1). The moment map J_{III} is defined by

$$J_{III}(x_1, x_2, y_1, y_2) := \mathcal{N}(\lambda) = \begin{pmatrix} \kappa t & 0 \\ 0 & -\kappa t \end{pmatrix} - \frac{\begin{pmatrix} x_1 y_1 + x_2 y_2 + \mu_1 & 2 \left(y_1 y_2 - \frac{\mu_1 \mu_2}{x_2^2} + \frac{\mu_2^2 x_1}{x_2^3} \right) \\ -2 x_1 x_2 & -x_1 y_1 - x_2 y_2 + \mu_1 \end{pmatrix}}{2\lambda} - \kappa t \frac{\begin{pmatrix} y_1 x_2 + \mu_2 & y_1^2 - \frac{\mu_2^2}{x_2^2} \\ -x_2^2 & -y_1 x_2 + \mu_2 \end{pmatrix}}{2\lambda^2}, \quad (2.23)$$

where μ_1, μ_2, κ are constants, $\kappa \neq 0$ and t is the deformation parameter. The isotropy group H_Y is again given by (2.14) and its action on M is generated by

$$a := \frac{1}{2}(x_1 y_1 + x_2 y_2). \quad (2.24)$$

The spectral Darboux coordinates for this case are

$$u = -\frac{\kappa t x_2}{2 x_1}, \quad v = \kappa t \frac{y_1 x_2 - \mu_2}{2 u^2} + \frac{x_1 y_1 + x_2 y_2 - \mu_1}{2 u}, \quad w = -x_1 x_2, \quad a = \frac{1}{2}(x_1 y_1 + x_2 y_2), \quad (2.25)$$

with ω again of the form (2.17). On the level set $a = a_0$ the symplectic form ω projects to

$$\omega_{\text{red}} = du \wedge dv. \quad (2.26)$$

The relevant Hamiltonian \mathcal{H}_{III} on M for this case is given by

$$\begin{aligned}\mathcal{H}_{III} &= \frac{1}{t} \text{res}_{\lambda=0} \text{tr}(\lambda \mathcal{N}^2(\lambda)) \\ &= -2\kappa^2 t y_1 x_2 + \frac{2}{t} \left(\frac{(x_1 y_1 - x_2 y_2)^2}{4} + \mu_1 \mu_2 \frac{x_1}{x_2} - \mu_2^2 \frac{x_1^2}{x_2^2} \right) + \frac{\mu_1^2}{2t}.\end{aligned}\quad (2.27)$$

The differential operator \mathcal{D}_t defining the isomonodromic deformation equation of \mathcal{D}_λ is obtained by setting $\sigma = -1/2$ in (1.16b), giving

$$\mathcal{D}_t = \frac{\partial}{\partial t} - \lambda \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix} + \frac{1}{2t} \begin{pmatrix} x_1 y_1 + x_2 y_2 + \mu_1 & 2 \left(y_1 y_2 - \frac{\mu_1 \mu_2}{x_2^2} + \frac{\mu_2^2 x_1}{x_2^3} \right) \\ -2 x_1 x_2 & -x_1 y_1 - x_2 y_2 + \mu_1 \end{pmatrix} - \frac{\kappa}{2\lambda} \begin{pmatrix} y_1 x_2 + \mu_2 & y_1^2 - \frac{\mu_2^2}{x_2^2} \\ -x_2^2 & -y_1 x_2 + \mu_2 \end{pmatrix}.\quad (2.28)$$

The corresponding Hamiltonian $\tilde{\mathcal{H}}_{III}$ on the reduced phase space is

$$\tilde{\mathcal{H}}_{III} = \frac{2}{t} (u^2 v^2 - 2\kappa t v u^2 + \mu_1 u v + \kappa \mu_2 t v + (2a_0 - \mu_1) t u) + \frac{\mu_1^2}{t} - 2\kappa^2 t \mu_2. \quad (2.29)$$

Taking into account the explicit t -dependence of the coordinates u, v defined in (2.25), Hamilton's equations for $\tilde{\mathcal{H}}_{III}$ are equivalent to the equation of the Painlevé transcendent P_{III} :

$$\ddot{u} = \frac{(\dot{u})^2}{u} - \frac{\dot{u}}{t} + \frac{1}{t}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}, \quad (2.30)$$

where the constants $\alpha, \beta, \gamma, \delta$ are given by

$$\alpha = -16a_0 + 8\mu_1 - 8\kappa\mu_1 - 8\kappa, \quad \beta = -4\kappa\mu_1\mu_2, \quad \gamma = 16\kappa^2, \quad \delta = -4\kappa^2\mu_2^2. \quad (2.31)$$

Painlevé IV. Again, $M = \mathbf{R}^4$ with symplectic form (2.1). The moment map J_{IV} is

$$\begin{aligned}J_{IV}(x_1, x_2, y_1, y_2) &:= \\ \mathcal{N}(\lambda) &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 2y_1 \\ -x_1 & 0 \end{pmatrix} + \frac{1}{2(\lambda - t)} \begin{pmatrix} -x_2 y_2 + \mu & -y_2^2 + \mu^2 x_2^{-2} \\ x_2^2 & x_2 y_2 + \mu \end{pmatrix},\end{aligned}\quad (2.32)$$

where μ is constant and t is the deformation parameter. The isotropy group H_Y is again given by (2.14) and its action on M is generated by

$$a = x_1 y_1 + \frac{x_2 y_2}{2}. \quad (2.33)$$

The spectral Darboux coordinates for this case are

$$u = \frac{x_2^2}{2x_1}, \quad v = \frac{x_2 y_2 + \mu}{2u}, \quad a = x_1 y_1 + \frac{x_2 y_2}{2}, \quad w = x_1, \quad (2.34)$$

with ω again of the form (2.17). The unreduced Hamiltonian \mathcal{H}_{IV} on M leading to the relevant isomonodromic deformation equation corresponding to P_{IV} is given by

$$\mathcal{H}_{IV} = \frac{1}{2} \text{res}_{\lambda=t} \text{tr}(\mathcal{N}(\lambda)^2) = -t x_2 y_2 + x_2^2 y_1 + \frac{1}{2} x_1 y_2^2 - \frac{\mu^2}{2} \frac{x_1}{x_2^2}. \quad (2.35)$$

The corresponding isomonodromic deformation operator \mathcal{D}_t is obtained by setting $\sigma = -1$ in (1.16b), giving

$$\mathcal{D}_t = \frac{\partial}{\partial t} + \frac{\begin{pmatrix} -x_2 y_2 + \mu & -y_2^2 + \mu^2 x_2^{-2} \\ x_2^2 & x_2 y_2 + \mu \end{pmatrix}}{2(\lambda - t)}. \quad (2.36)$$

Fixing the level set $a = a_0$ and quotienting by the H_Y -action, the reduced symplectic form is again of the form (2.26), while the reduced Hamiltonian is

$$\tilde{\mathcal{H}}_{IV} = -t(2uv - \mu) + (2a_0 + \mu)u - 2u^2 v + uv^2 - \mu v. \quad (2.37)$$

Hamilton's equations for $\tilde{\mathcal{H}}_{IV}$ imply the equation of motion

$$\ddot{u} = \frac{(\dot{u})^2}{2u} + 6u^3 + 8tu^2 + 2(t^2 - \alpha)u + \frac{\beta}{u}, \quad (2.38)$$

where

$$\alpha = 2a_0 + 1, \quad \beta = -\frac{\mu^2}{2}. \quad (2.39)$$

The standard normalization for P_{IV} (cf. [I]) is obtained by making the change

$$\tilde{u} = 2u. \quad (2.40)$$

Painlevé V. Again $M = \mathbf{R}^4$ with symplectic form (2.1). The moment map J_V is defined by

$$J_V(x_1, x_2, y_1, y_2) := \mathcal{N}(\lambda) = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} + \frac{\begin{pmatrix} -x_1 y_1 - \mu_1 & -y_1^2 + \mu_1^2 x_1^{-2} \\ x_1^2 & x_1 y_1 - \mu_1 \end{pmatrix}}{2\lambda} + \frac{\begin{pmatrix} -x_2 y_2 - \mu_2 & -y_2^2 + \mu_2^2 x_2^{-2} \\ x_2^2 & x_2 y_2 - \mu_2 \end{pmatrix}}{2(\lambda - 1)} \quad (2.41)$$

where μ_1, μ_2 are constants and t is the deformation parameter. The isotropy group H_Y is again given by (2.14) and its action on M is generated by

$$a := \frac{1}{2}(x_1 y_1 + x_2 y_2). \quad (2.42)$$

The spectral Darboux coordinates for this case are given by

$$u = \frac{x_1^2}{x_1^2 + x_2^2}, \quad v = \frac{1}{2} \left(\frac{x_1 y_1 + \mu_1}{u} + \frac{x_2 y_2 + \mu_2}{u - 1} \right), \quad (2.43)$$

$$a = \frac{1}{2}(x_1 y_1 + x_2 y_2), \quad w = x_1^2 + x_2^2.$$

The unreduced Hamiltonian \mathcal{H}_V on M leading to the relevant isomonodromic deformation equation is given by

$$\begin{aligned}\mathcal{H}_V &= \frac{1}{2t} \text{res}_{\lambda=1} \text{tr}(\mathcal{N}(\lambda)^2) - \frac{1}{16t^2} [\text{res}_{\lambda=1} \text{tr}(\mathcal{N}(\lambda)^2) + \text{res}_{\lambda=0} \text{tr}(\mathcal{N}(\lambda)^2)]^2 \\ &= -\frac{1}{4t} (x_1^2 + x_2^2)(y_1^2 + y_2^2) + \frac{1}{4t} \left(\mu_1^2 \frac{x_2^2}{x_1^2} + \mu_2^2 \frac{x_1^2}{x_2^2} \right) - x_2 y_2.\end{aligned}\quad (2.44)$$

The differential operator \mathcal{D}_t defining the corresponding isomonodromic deformation equation is obtained by setting $\sigma = 0$ in (1.16b), giving

$$\mathcal{D}_t = \frac{\partial}{\partial t} - \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} - \frac{1}{2t} \begin{pmatrix} 0 & -y_1^2 - y_2^2 + \mu_1 \frac{y_1}{x_1} + \mu_2 \frac{y_2}{x_2} \\ x_1^2 + x_2^2 & 0 \end{pmatrix}. \quad (2.45)$$

Reducing at the level set $a = a_0$, the reduced Hamiltonian $\tilde{\mathcal{H}}_V$ becomes

$$\tilde{\mathcal{H}}_V = \frac{1}{t} (v^2 u^2 - v^2 u - 2t v u^2 + u v (2t - \mu_1 - \mu_2) + u t (\mu_1 + \mu_2 + 2a_0) + \mu_1 v + \mu_1 \mu_2). \quad (2.46)$$

The equations of motion generated by $\tilde{\mathcal{H}}_V$ are

$$\begin{aligned}\frac{d^2 u}{dt^2} &= \left(\frac{1}{2u} + \frac{1}{2(u-1)} \right) \left(\frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} - \alpha \frac{u}{t^2(u-1)} - \beta \frac{u-1}{t^2 u} \\ &\quad - \gamma \frac{u(u-1)}{t} - \delta u(u-1)(2u-1),\end{aligned}\quad (2.47)$$

where

$$\alpha = \frac{\mu_2^2}{2}, \quad \beta = -\frac{\mu_1^2}{2}, \quad \gamma = 4a_0 + 2, \quad \delta = 2. \quad (2.48)$$

The standard form of P_V (cf. [I]) is obtained through the transformation

$$\tilde{u} = \frac{u}{u-1}. \quad (2.49)$$

Painlevé VI. In this case, we begin with $M = \mathbf{R}^6$, with standard symplectic form

$$\omega = \sum_{i=1}^3 dx_i \wedge dy_i. \quad (2.50)$$

The moment map $J_{VI} : \mathbf{R}^6 \rightarrow \tilde{\mathfrak{g}}_R^*$ is given by

$$\begin{aligned}J_{VI}(x_1, y_1, x_2, y_2, x_3, y_3) &:= \\ \mathcal{N}(\lambda) &= \frac{\begin{pmatrix} -x_1 y_1 - \mu_1 & -y_1^2 + \mu_1^2 x_1^{-2} \\ x_1^2 & x_1 y_1 - \mu_1 \end{pmatrix}}{2\lambda} + \frac{\begin{pmatrix} -x_2 y_2 - \mu_2 & -y_2^2 + \mu_2^2 x_2^{-2} \\ x_2^2 & x_2 y_2 - \mu_2 \end{pmatrix}}{2(\lambda-1)} + \frac{\begin{pmatrix} -x_3 y_3 - \mu_3 & y_3^2 - \mu_3^2 x_3^{-2} \\ -x_3^2 & x_3 y_3 - \mu_3 \end{pmatrix}}{2(\lambda-t)},\end{aligned}\quad (2.51)$$

where μ_1, μ_2 and μ_3 are constants and t is the deformation parameter. In this case $Y = 0$ so that the isotropy group H_Y is given by

$$H_Y = Sl(2). \quad (2.52)$$

The action of H_Y on M is generated by the moment map

$$J_{Sl(2)} := \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2), \quad (2.53)$$

where

$$a = \frac{1}{2} \sum_{i=1}^3 x_i y_i \quad (2.54a)$$

$$b = -\frac{1}{2}(y_1^2 + y_2^2 - y_3^2) + \frac{\mu_1^2}{2x_1^2} + \frac{\mu_2^2}{2x_2^2} - \frac{\mu_3^2}{2x_3^2} \quad (2.54b)$$

$$c = -\frac{1}{2}(x_1^2 + x_2^2 - x_3^2). \quad (2.54c)$$

The spectral invariants are all invariant under the action of H_Y , so their Hamiltonian flows leave invariant the level set

$$b = c = 0, \quad (2.55)$$

which we choose as the “unreduced” phase space \widetilde{M} for P_{VI} . On \widetilde{M} we may again define spectral Darboux coordinates by

$$\begin{aligned} u &= \frac{tx_1^2}{w}, \quad w = (1+t)x_1^2 + tx_2^2 - x_3^2 \\ v &= \frac{1}{2} \left(\frac{x_1 y_1 - \mu_1}{u} + \frac{x_2 y_2 - \mu_2}{u-1} + \frac{x_3 y_3 - \mu_3}{u-t} \right), \end{aligned} \quad (2.56)$$

with a defined by (2.54a). The symplectic form $\tilde{\omega}$ on \widetilde{M} is expressed in terms of these coordinates by

$$\tilde{\omega} := \omega|_{\widetilde{M}} = d \ln(w) \wedge da + du \wedge dv. \quad (2.57)$$

The unreduced Hamiltonian \mathcal{H}_{VI} on M is given by

$$\begin{aligned} \mathcal{H}_{VI} &= \frac{1}{2} \text{res}_{\lambda=t} \text{tr}(\mathcal{N}(\lambda)^2) \\ &= \frac{1}{4t} \left[(x_1 y_3 + x_3 y_1)^2 - \mu_1^2 \frac{x_3^2}{x_1^2} - \mu_3^2 \frac{x_2^2}{x_3^2} + 2\mu_2 \mu_3 \right] \\ &\quad + \frac{1}{4(t-1)} \left[(x_2 y_3 + x_3 y_2)^2 - \mu_2^2 \frac{x_3^2}{x_2^2} - \mu_3^2 \frac{x_2^2}{x_3^2} + 2\mu_2 \mu_3 \right]. \end{aligned} \quad (2.58)$$

The differential operator \mathcal{D}_t defining the isomonodromic deformation equation corresponding to \mathcal{H}_{VI} is obtained by setting $\sigma = -1$ in (1.16b), giving

$$\mathcal{D}_t = \frac{\partial}{\partial t} + \frac{\begin{pmatrix} -x_3 y_3 - \mu_3 & y_3^2 - \mu_3^2 x_3^{-2} \\ -x_3^2 & x_3 y_3 - \mu_3 \end{pmatrix}}{2(\lambda - t)}. \quad (2.59)$$

Restricting to the invariant symplectic manifold \widetilde{M} , and reducing at the level set $a = a_0$ by the 1-parameter group generated on \widetilde{M} by a , the reduced symplectic form is again

$$\omega_{\text{red}} = du \wedge dv \quad (2.60)$$

and the reduced Hamiltonian is

$$\begin{aligned} \tilde{\mathcal{H}}_{VI} = & \frac{1}{t(t-1)} [u(u-1)(u-t)v^2 + v(\mu_1(u-1)(u-t) + \mu_2 u(u-t) + \mu_3 u(u-1)) \\ & + \mu_1 \mu_2 (u-t) + \mu_2 \mu_3 u + \mu_1 \mu_3 (u-1) + (t-u)K], \end{aligned} \quad (2.61)$$

where

$$K = a_0^2 + \frac{1}{4} \left(\sum_{i=1}^3 \mu_i \right)^2 - \frac{1}{2} \sum_{i=1}^3 \mu_i^2. \quad (2.62)$$

As in the case of P_{III} , the explicit t -dependence of the coordinate functions u, v defined in (2.56) must be taken into account when computing Hamilton's equations. The t derivatives implied by this explicit dependence are given by

$$\begin{aligned} u_t &= \frac{u(u-1)}{t(t-1)} \\ v_t &= \frac{-v(2u-1) + a_0 - \frac{1}{2}(\mu_1 + \mu_2 + \mu_3)}{t(t-1)} \end{aligned} \quad (2.63)$$

Inserting this into Hamilton's equations for $\tilde{\mathcal{H}}_{VI}$ and eliminating v gives P_{VI} as the equation of motion:

$$\begin{aligned} \frac{d^2 u}{dt^2} = & \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t} \right) \left(\frac{du}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t} \right) \frac{du}{dt} \\ & + \frac{u(u-1)(u-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{u^2} + \gamma \frac{t-1}{(u-1)^2} + \delta \frac{t(t-1)}{(u-t)^2} \right), \end{aligned} \quad (2.64)$$

where

$$\alpha = 2a_0^2 + 2a_0 + \frac{1}{2}, \quad \beta = -\frac{1}{2}\mu_1^2, \quad \gamma = \frac{1}{2}\mu_2^2, \quad \delta = -\frac{1}{2}\mu_3^2 + \frac{1}{2}. \quad (2.65)$$

3. Discussion

With the exception of P_{IV} , the isomonodromic deformations derived above are of the same form as those given in [JM]. The new element here is the fact that these systems are obtained as nonautonomous Hamiltonian systems through moment map embeddings into loop algebras, with the Hamiltonians chosen from the ring of spectral invariants. Moreover, the same “spectral Darboux coordinates” which are used to linearize the flows in the autonomous case lead here, after suitable reductions, to the canonical coordinates (u, v) , determining the Hamiltonian form of the Painlevé equations. The resulting Hamiltonians coincide essentially with those in [Ok] (after the transformations (2.40) and (2.49)).

Prior to reduction, all the Hamiltonians (2.9), (2.18), (2.27), (2.35), (2.44) and (2.58) have the general form

$$\mathcal{H} = \frac{1}{2} \sum_{i,j=1}^m g^{ij}(y_i + A_i(x))(y_j + A_j(x)) + V(x), \quad (3.1)$$

where $m = 2$ for $P_I - P_V$ and $m = 3$ for P_{VI} , for suitably defined symmetric contravariant tensor, covector and scalar fields, g^{ij} , A_i , v , respectively. If the tensors g^{ij} were all non-singular, we could interpret these as simple magneto-mechanical systems with metric g_{ij} given by the covariant tensor inverse to g^{ij} , the A_i ’s interpreted as vector potentials and the $V(x)$ as scalar potentials. This is indeed the case for P_V and P_{VI} , although the further symplectic constraints (2.55) must be added in the case P_{VI} . For P_V , Eq. (2.44) gives the following metric tensor, vector potential and scalar fields:

$$\begin{aligned} ds^2 &= \sum_{i,j=1}^2 g_{ij} dx_i dx_j = -\frac{2t}{x_1^2 + x_2^2} (dx_1^2 + dx_2^2) \\ A &= \sum_{i=1}^2 A_i dx_i = \frac{2tx_2}{x_1^2 + x_2^2} dx_2 \\ V(x) &= \frac{1}{4t} \left(\mu_1^2 \frac{x_2^2}{x_1^2} + \mu_2^2 \frac{x_1^2}{x_2^2} \right) + \frac{tx_2^2}{x_1^2 + x_2^2} + C. \end{aligned} \quad (3.2)$$

For $P_I - P_{IV}$ the tensors g^{ij} are singular.

The moment map embeddings characterizing the orbits and choice of Hamiltonians for the cases P_V and P_{VI} are well understood within the broader framework [AHH3, HW] of moment map embeddings in rational Ad^* orbits of loop algebras. For the other cases, the moment maps and spectral invariant Hamiltonians have here been constructed case by case, chosen to suit the known singularity structure of the corresponding isomonodromic deformation equations. A general formulation, based on moment maps to loop algebras, allowing higher order poles at $\lambda = \infty$ as well as at finite points and a characterization of those spectral invariant Hamiltonians which generate isomonodromic deformations in the general case has yet to be developed.

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